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B. Chang

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The P_n reduction factors, obtained by Haan[1] and by Phillion et.al.[2], for the luminance, an integral involving the specific intensity, in a spherical hohlraum were derived independently of the specific intensity determined by transport. Their results are plausible but unconvincing because of this disconnect. The purpose of this note is to derive the reduction factors from the specific intensity ψ which solves the transport equation

$$(0.1)\quad \mu\frac{\partial\psi}{\partial r}+\frac{\sqrt{1-\mu^2}\,\cos\omega}{r}\frac{\partial\psi}{\partial\theta}+\frac{1-\mu^2}{r}\frac{\partial\psi}{\partial\mu}-\frac{\sqrt{1-\mu^2}\,\cos\theta\,\sin\omega}{r\sin\theta}\frac{\partial\psi}{\partial\omega}=0\,,$$

for the boundary condition

$$(0.2)\qquad \psi(r=b,\cos\theta,\mu,\cos\omega)=\psi_b(\cos\theta,\mu,\cos\omega),\qquad \mu<0\,,$$

in a spherical cavity of radius b under the assumption of azimuthal symmetry. The variables of (0.1), displayed in Fig.2.1, are: r is the distance variable of the spherical coordinate system, θ is the polar angle of the coordinate system, μ is the dot product of the photon's direction and $\mathbf{e}_r$, and $\sqrt{1-\mu^2}\cos\omega$ is the dot product of the photon's direction and $\mathbf{e}_\theta$.

The solution to (0.1), obtained in [3], is

$$(0.3)\qquad \psi(r,\cos\theta,\mu,\cos\omega)=\psi_b(\cos\theta_0,\mu_0,\cos\omega_0)$$

where

$$(0.4)\qquad \mu_0=-\sqrt{1-\frac{r^2}{b^2}(1-\mu^2)}\,,$$

$$(0.5)\quad \cos\theta_0=\frac{r}{b}\cos\theta-\Big(\mu\cos\theta-\sqrt{1-\mu^2}\,\sin\theta\cos\omega\Big)\left(\frac{r}{b}\mu+\sqrt{1-\frac{r^2}{b^2}(1-\mu^2)}\right),$$

$$(0.6)\quad \cos\omega_0=\frac{\frac{r}{b}\sin\theta\cos\omega-\Big(\mu\sin\theta\cos\omega+\sqrt{1-\mu^2}\,\cos\theta\Big)\Big(\frac{r}{b}\mu+\sqrt{1-\frac{r^2}{b^2}(1-\mu^2)}\Big)}{\sin\theta_0}.$$

The factor $\sin\theta_0$ which is in the denominator of the term on the right hand side (rhs) of (0.6) is an expression for $\sqrt{1-\cos^2\theta_0}$ where $\cos\theta_0$ is given by (0.5).

We can verify that (0.3) solves (0.1) by substitution. In particular, the substitution of (0.3) into the left hand side (lhs) of (0.1) yields zero, because the functions on the rhs of (0.3), the arguments of ψ_b, are in the null space of the operator on the lhs of (0.1). For example, the function $\cos\theta_0$ of (0.5) is in the null space of the operator, because

$$\mu\frac{\partial\cos\theta_0}{\partial r}+\frac{\sqrt{1-\mu^2}\cos\omega}{r}\frac{\partial\cos\theta_0}{\partial\theta}+\frac{1-\mu^2}{r}\frac{\partial\cos\theta_0}{\partial\mu}-\frac{\sqrt{1-\mu^2}\cos\theta\sin\omega}{r\sin\theta}\frac{\partial\cos\theta_0}{\partial\omega}=0\,.$$

The substitutions of μ_0 and $\cos\omega_0$ into the lhs of (0.1) also yield zeros. Moreover (0.3) satisfies (0.2), because $\mu_0 = -|\mu|$, $\cos\theta_0 = \cos\theta$, and $\cos\omega_0 = \cos\omega$ when r is set equal to b in (0.4), (0.5), and (0.6), respectively. If the boundary condition (0.2) is independent of both θ and ω, which is the spherically symmetric case, then (0.3) simplifies to the result obtained by Lathrop (the equation above eq.(4) of [4]).

Even if the boundary condition is not spherically symmetric, it can however be represented as an expansion in Legendre polynomials

$$\psi_b(\cos\theta, \mu, \cos\omega) = \sum_{n=0}^{\infty} \epsilon_n(\mu, \cos\omega)\, P_n(\cos\theta)\ , \tag{0.7}$$

where $\epsilon_n(\mu, \cos\omega)$, 'the amplitude of the P_n mode', is given by

$$\epsilon_n(\mu, \cos\omega) = \frac{2n+1}{2} \int_{-1}^{1} d(\cos\theta)\ P_n(\cos\theta)\ \psi_b(\cos\theta, \mu, \cos\omega)\ . \tag{0.8}$$

Moreover, the Legendre expansion enables the analytical solution (0.3) to be expressed as

$$\psi(r, \cos\theta, \mu, \cos\omega) = \sum_{n=0}^{\infty} \epsilon_n(\mu_0, \cos\omega_0)\, P_n(\cos\theta_0)\ . \tag{0.9}$$

1. The calculation of the luminance. The calculation of the luminance is a post-processing procedure for obtaining physical information from ψ, not a method for obtaining ψ. For example, the luminance $F_b(\cos\theta)$ from the boundary is obtained from $\psi_b(\cos\theta, \mu, \cos\omega)$ by

$$F_b(\cos\theta) = -\int_{-\pi}^{\pi} d\omega \int_{-1}^{0} d\mu\ \ \mu\ \ \psi_b(\cos\theta, \mu, \cos\omega)\ , \tag{1.1}$$

which is the flux flowing into the cavity from the boundary.

If ψ_b is independent of μ and $\cos\omega$, which is the assumption from hereon, then $\psi_b(\cos\theta)$ can be taken out of the integral to yield

$$F_b(\cos\theta) = \ \pi\ \psi_b(\cos\theta)\ . \tag{1.2}$$

Since $\psi_b(\cos\theta)$ is independent of μ and $\cos\omega$ which means that ϵ_n of (0.8) is also independent of μ and $\cos\omega$, then the Legendre decomposition of $F_b(\cos\theta)$ can be obtained by substituting (0.7) into (1.1) to yield

$$F_b(\cos\theta) = \sum_{n=0}^{\infty} \epsilon_n\, \pi\ P_n(\cos\theta)\ . \tag{1.3}$$

The quantity $\epsilon_n\pi$ is said to be the amplitude of the P_n mode of the luminance emitted from the boundary.

The calculation of $F(r, \cos\theta)$ the luminance at the point (r, θ) in the spherical cavity is also a post-processing procedure; it is obtained by

$$F(r, \cos\theta) = -\int_{-\pi}^{\pi} d\omega \int_{-1}^{0} d\mu\ \ \mu\ \ \psi(r, \cos\theta, \mu, \cos\omega)\,, \tag{1.4}$$

where $\psi(r, \cos\theta, \mu, \cos\omega)$, the solution to (0.1), is given by (0.9). The substitution of (0.9) into (1.4) under the assumption that ϵ_n is independent of μ and ω yields

$$F(r, \cos\theta) = -\sum_{n=0}^{\infty} \epsilon_n \int_{-\pi}^{\pi} d\omega \int_{-1}^{0} d\mu \;\; \mu \;\; P_n(\cos\theta_0) \tag{1.5}$$

The double integral in (1.5) can be reduced to an integral by using the addition theorem of spherical harmonics [5] to express $P_n(\cos\theta_0)$ as a sum of factors which can be integrated analytically with respect to ω. Let $\eta \equiv {}^{r}/_{b}$ denote the ratio of r and the radius of the cavity, and

$$s \;\equiv\; \eta\,\mu + \sqrt{1 - \eta^2(1 - \mu^2)} \tag{1.6}$$

denote a ω-independent variable. These variables enable us to express $\cos\theta_0$ of (0.5) as

$$\cos\theta_0 = (\eta - s\,\mu)\,\cos\theta + \left(s\,\sqrt{1 - \mu^2}\right)\sin\theta\cos\omega\,. \tag{1.7}$$

Since $(\eta - s\,\mu)$ the coefficient multiplying $\cos\theta$, and $\left(s\,\sqrt{1 - \mu^2}\right)$ the coefficient multiplying $\sin\theta\cos\omega$ are related by

$$(\eta - s\,\mu)^2 + \left(s\,\sqrt{1 - \mu^2}\right)^2 = 1, \tag{1.8}$$

then (1.7) can be written as

$$\cos\theta_0 \;=\; \cos\chi\,\cos\theta \;+\; \sin\chi\,\sin\theta\,\cos\omega\;, \tag{1.9}$$

where

$$\cos\chi \equiv (\eta - s\,\mu)\,, \quad \text{and} \tag{1.10}$$

$$\sin\chi \equiv \left(s\,\sqrt{1 - \mu^2}\right). \tag{1.11}$$

Therefore, the addition theorem of spherical harmonics [5] enables the Legendre polynomial $P_n(\cos\theta_0)$ of (1.5) to be expressed as

$$P_n(\cos\theta_0) = P_n(\cos\chi)P_n(\cos\theta) + 2\sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\chi)P_n^m(\cos\theta)\cos m\omega. \tag{1.12}$$

Since the integral of $\cos m\omega$ with respect to ω from $-\pi$ to π vanishes, then the substitution of (1.12) into (1.5) followed by the substitution of (1.10) into the result yields

$$F(r, \cos\theta) = \sum_{n=0}^{\infty} \epsilon_n\,\pi\,P_n(\cos\theta)\,f_n(\eta)\;, \tag{1.13}$$

where

$$f_n(\eta) \;\equiv\; -2\int_{-1}^{0} d\mu\;\mu\;P_n\Big(\,\eta(1-\mu^2) - \mu\sqrt{1 - \eta^2(1-\mu^2)}\,\Big)\;. \tag{1.14}$$

Since $\epsilon_n\pi$ is the amplitude of the n$^{\text{th}}$ Legendre polynomial mode of luminance emitted from the boundary (see (1.3)), and $\epsilon_n\pi f_n(\eta)$ is the amplitude of the mode illuminating the spherical surface of radius r, then we can interpret $f_n(\eta)$ as the P_n reduction factor obtained by solving (0.1) for the boundary condition $\psi_b(\cos\theta) = \sum_{n=0}^{\infty} \epsilon_n P_n(\cos\theta)$.

2. The relationship of $f_n(\eta)$ to Haan's formula for the reduction factor. We shall now derive Haan's formula from (1.14) by changing the integration variable from μ to

$$z = \left(\sqrt{1-\eta^2(1-\mu^2)} \;+\; \eta\,\mu\right)^2 . \tag{2.1}$$

The integration limits which correspond to $\mu = -1$, and $\mu = 0$ are $z = (1-\eta)^2$, and $z = 1-\eta^2$, respectively. In order to simplify the derivation, we need the inverse function to (2.1)

$$\mu = \frac{z+\eta^2-1}{2\,\eta\, z^{1/2}} . \tag{2.2}$$

The substitution of (2.2) into the argument of the Legendre polynomial of (1.14) gives

$$P_n\left(\eta(1-\mu^2) - \mu\sqrt{1-\eta^2(1-\mu^2)}\right) \;=\; P_n\left(\frac{1+\eta^2-z}{2\,\eta}\right) . \tag{2.3}$$

Now the differential of (2.2) is

$$d\mu = \frac{1}{4\,\eta\, z^{1/2}}\left(1 - \frac{\eta^2-1}{z}\right) dz . \tag{2.4}$$

Therefore, we have

$$-2\mu\, d\mu \;=\; -\frac{z+\eta^2-1}{4\eta^2\, z}\left(1-\frac{\eta^2-1}{z}\right) dz \;=\; \frac{1}{4\eta^2}\left(\frac{(1-\eta^2)^2}{z^2} - 1\right) dz \tag{2.5}$$

by (2.2), and (2.4). The substitutions of (2.3) and (2.5) into (1.14) give

$$f_n(\eta) \;=\; \frac{1}{4\,\eta^2}\int_{(1-\eta)^2}^{1-\eta^2} \left(\frac{(1-\eta^2)^2}{z^2} - 1\right)\; P_n\left(\frac{1+\eta^2-z}{2\,\eta}\right)\; dz , \tag{2.6}$$

the result derived by Haan (eq.(13) of [1]).

The results of the integrals in (2.6), obtained by the algorithm provided in [1], are displayed in Fig.2.2 as circles. On the other hand, the results of the double integrals in (1.5) for $\theta = 0$, converged to 6 significant digits, divided by π, and presented as dots in the figure, are in perfect agreement with the data presented as circles. It was difficult to determine (2.6) accurately for small η, because the integrals, as noted by Haan, are sensitive to round-off in this region. Therefore the results of (2.6) for $\eta = 0$ are omitted from the figure. We also note that the integrals in (1.14) are more stable than the integrals in (2.6). A four-point Gaussian approximation to the integrals in (1.14) yields 6 digits of accuracy for $n = 0, \ldots, 6$, and $0 \le \eta \le 1$.

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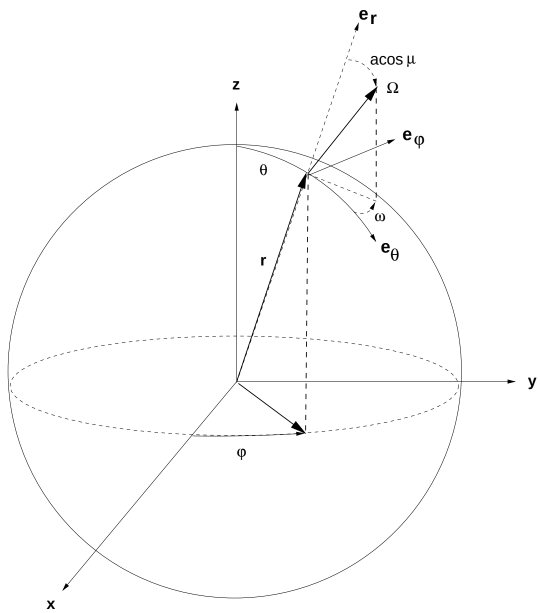

FIG. 2.1. *The position vector* $\mathbf{r}$*, and the direction vector* $\mathbf{\Omega}$ *in spherical coordinates.*

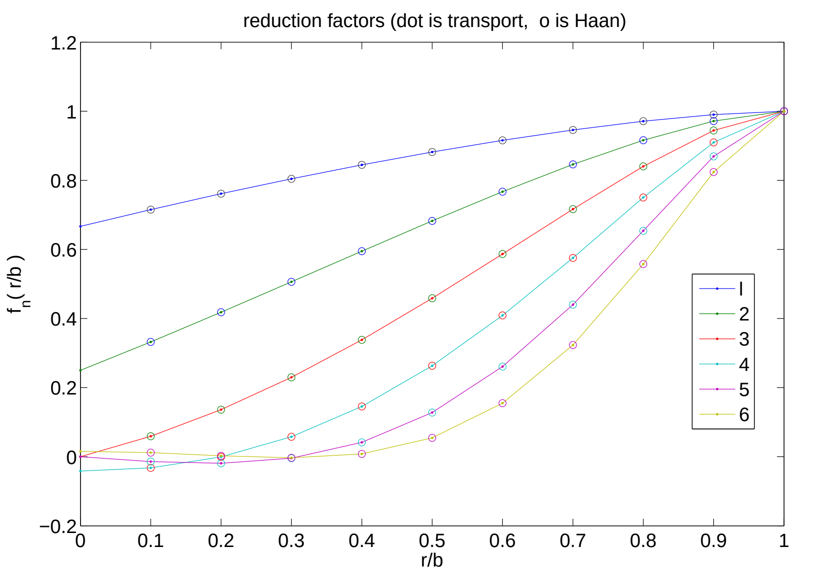

FIG. 2.2. *The integrals of (2.6)* (o)*, the double integrals of (1.5) (divided by* π*) for* $\theta = 0$ $(\cdot)$*, and their* P_n *mode numbers listed in the legend.*

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